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JULIA SETS IN THE QUATERNIONS

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Abstract—Recent mathematical work on the dynamics of complex analytic functions has given rise to a new subject matter for computer graphics. The combination of mathematical theory and computer graphics has resulted in new insight into the nature of some of the simplest of mathematical objects, second-degree polynomials. Most of that work has focused on the possibilities within the two-dimensional complex plane. This article shows how these investigations may be extended to higher dimensions, resulting in fractals that naturally reside in the 4-dimensional quaternions. Particular attention is paid to the formula $ax^2 + b$. A method is given for obtaining various interconnection patterns for the Julia sets in 4-space, and the results are displayed in 3-D computer graphics.

INTRODUCTION

Quadratic polynomials are usually presented early in elementary algebra courses, and illustrated using a parabola. One learns how to calculate roots, to locate the focus and directrix. Since all parabolas look about the same, the subject is easily treated in one or two lectures, followed by generalizations to higher degree polynomials. This subject, at least, is one which apt students can understand completely, and use as a simple model for the relationship between algebra and geometry. Or so it seems.

But now look at the pictures illustrating this article. These pictures are directly derived from quadratic polynomials; in fact can be regarded as pictures *of* quadratic polynomials. These shapes of endless detail are in many respects more naturally associated with the polynomials than are parabolas. One needs more than a pen and graph paper to generate such drawings. The quadratic mapping reveals its inexhaustible content only when examined by the computer.

Several studies have appeared in recent years demonstrating the dramatic visual effects obtainable from applying two-dimensional computer graphics to complex polynomials. Fascinating as such pictures are, they are only slim fragments compared to the three- or four-dimensional physical reality. Quadratics do in fact reside in higher dimensions, and we present one such extension in this study.

We shall show how the 4-dimensional quaternion algebra can be used to define structures possessing complex patterns of infinitely repeating geometric structure. We do not have complete control over the structure, comparable to the way a sculptor can prescribe the topology and texture of the object being created. We can, however, to a limited extent define the interconnection patterns of these shapes, and will show how such controls can be exploited.

An underlying theme in this study is the presence of endlessly repeating geometric patterns. The geometric objects revealed through these techniques are called fractals, and satisfy Mandelbrot's definition [5]. Mandelbrot asserted that fractals are the Geometry of Nature. In the illustrations of Julia sets in this article we see that the universe of mathematical shapes does not differ in that respect from the real world: Typical Julia sets are fractals; only rarely do we encounter the smooth objects of Euclidean geometry.

DYNAMICS IN THE COMPLEX PLANE

Points in a plane can be represented with two real coordinates. But about three centuries ago it was discovered that much simplicity and conceptual understanding is gained by recognizing the pair of real numbers as a kind of number in its own right, a "complex number." Instead of the pair (x, y) representing a point in the plane, the single entity $x + y\sqrt{-1}$ is used. This required introduction of a new "imaginary" number, $i = \sqrt{-1}$. Unlike real numbers, the number i does not represent distance along a straight line. Instead, i may be treated as a displacement in a direction perpendicular to the real number line. Using this new kind of number, all points in a plane are then regarded as "complex" numbers. The point with coordinates (x, y) is regarded as the complex number $x + iy$.

Like real numbers, complex numbers can be added and subtracted, multiplied and divided (except that we still are not allowed to divide by zero.) The complex numbers contain the real numbers as one line (the "real axis") in the complex plane. The rules of complex arithmetic are easily expressed in terms of the arithmetic of real numbers:

The sum of $x + iy$ and $v + iw$ is $(x + v) + i(y + w)$.
Their product is $(xy - vw) + i(xw + yv)$.

By thinking of points in the plane as complex numbers, we can manipulate geometry with formulas. Consider for example a quadratic polynomial

$$p(x) = x^2 - 1.$$

This formula determines a value, $p(x)$, associated with a number x. It works equally well whether x is a complex or real number. We can regard it as a geometric rule that, for each point z in the plane associates a point $p(z)$, also in the complex plane.

Complex derivatives

Many of the operations one performs with formulas make sense whether the formula is applied to real or complex numbers. One example we shall use later is the notion of a complex derivative of a polynomial. Derivatives are usually defined (in calculus class) as the slope of a line tangent to a graph. When we express that definition as a limit

$$\lim_{h \to 0} \frac{p(x+h) - p(x)}{h}$$

we see that the limit can make sense even if x and h are complex numbers and if $p(x)$ takes complex values. This derivative of a complex polynomial is then another complex polynomial, and can be computed by the power rule:

$$d\frac{ax^n}{dx} = anx^{n-1}.$$

The mathematical theory of complex analytic functions was developed during the 19th century. These functions have complex derivatives (as in the above limit) where they are defined. They include, for example, polynomials and the exponential function, but not the conjugation function $f(x + iy) = (x - iy)$.

Geometrically, what makes analytic functions so special is the fact that they are "conformal mappings." To explain this, we need to think of formulas as mappings, in the sense that a formula $f(x)$ not only tells what f does to a point x, it also gives a way of taking points near x and associating them with points near $f(x)$. Geometrically, we can think of f as stretching or otherwise distorting a piece of a plane near x as it pastes it onto the plane of $f(x)$. (This is illustrated in Fig. 1.)

In this context, conformality means that the mapping has a particularly nice property: It preserves angles. If two lines, crossing at x, form an angle θ, then the mapping f may distort the lines so that they are curves instead of straight lines, but the resulting curves still meet at the same angle θ. Complex analytic functions (and polynomials in particular) have the property of being conformal everywhere they are defined, except at the places where the complex derivative $f'(x)$ vanishes.

What is dynamics?

The subject of dynamics is concerned with what happens to a physical or geometric system over time, when it is subjected to a force, or undergoes some kind of manipulation. For example the motion of planets about the sun can be modelled as a dynamical system in which the planets move according to Newton's laws. These laws provide a set of rules that one can use to compute the position and velocity of the planets tomorrow, if one is given their position and velocity today. With enough effort we could derive a formula that would approximate what the passage of a fixed time (i.e., one day) does to the solar system. Similarly, a complex polynomial $p(x)$ can be regarded as a rule for moving points in the complex plane. If an object is positioned at z_0 at time $t = 0$, then $z_1 = p(z_0)$ will be its position at time $t = 1$, $p(p(z_0)) = p(z_1)$ will be its position at $t = 2$, etc. Here of course the rule is completely nonphysical, having nothing to do with Newton's laws.

Dynamics is concerned more with the long-term behavior of a dynamical system than the explicit rule defining the change from one time to the next. We would prefer to know whether the earth will eventually fall into the sun than where the earth will be tomorrow. Similarly, with a polynomial $p(x)$, one would like to understand the limiting effect of "iterating" or composing the polynomial with itself. To establish some notation, let $f^{(n)}(x)$ denote the result of applying (or composing) the function $f(x)$ n times to the starting value x:

$$f^{(n)}(x) = f(f(\cdots(f(x))\cdots)).$$

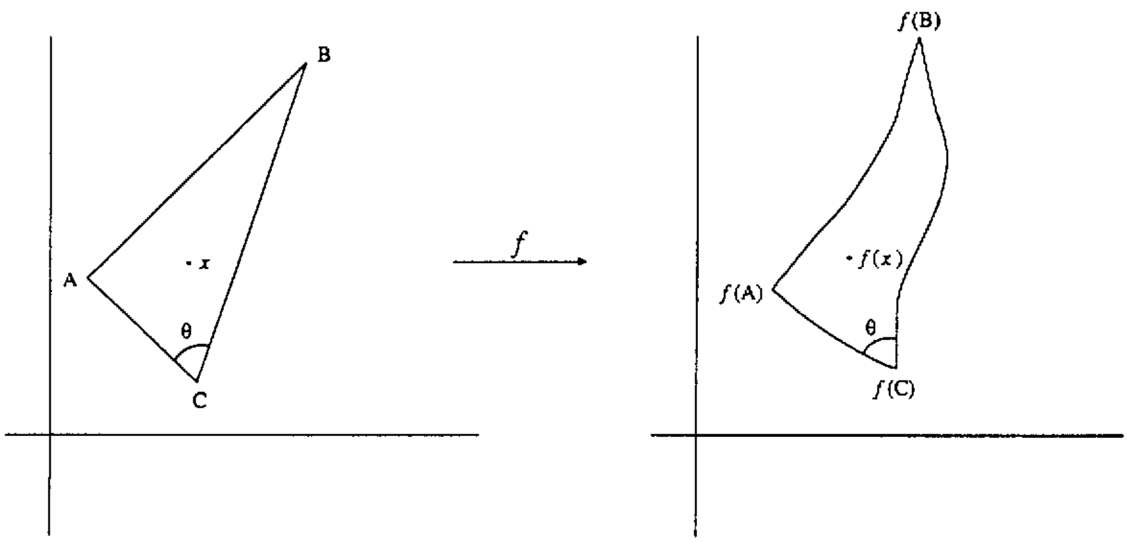


Fig. 1. Conformal mapping. A complex analytic function may be regarded as a rule that moves a portion of the complex plane to another portion. This motion may cause some distortion, but angles are preserved.

What sort of thing can happen to $f^{(n)}(z_0)$ as n becomes large? We give some examples, to motivate some definitions. If $f(z) = z^2$ and we start with $z_0 = 2$, it is easy to compute that $f^{(n)}(z_0)$ becomes arbitrarily large, or converges to infinity. Similarly, taking $z_0 = \frac{1}{2}$, successive iterates rapidly approach zero. In general, if z_0 is within a circle of radius 1 of the origin, then $f^{(n)}(z_0)$ converges to zero, and it converges to infinity whenever z_0 is outside that circle. This illustrates the phenomenon of "attraction:" Zero is "attractive" in the sense that points nearby zero are moved closer when the formula is applied. Infinity is also said to be attractive in this context, since complex numbers far from the origin are moved further by the mapping.

There is also a converse notion of "repulsion." Consider how the transformation $f(x) = x^2$ acts on the two numbers 1 and 1.01. Successive iterations of 1 do not change its position; however, iterating 1.01, we obtain successively 1.0201, 1.0406 $\cdots$, 1.08 $\cdots$. In fact, every point nearby to 1, other than 1 itself, moves away from 1 when the formula is applied. We say that 1 is "repulsive" or a "repeller" for the function f.

Another concept worth defining occurs in the above example. Note that repeated applications of the formula f do not move the points 0 or 1. These points are called "fixed points" of f. More generally, it can occur that after several iterations a point returns to its starting position. To be precise, suppose that $f^{(k)}(z_0) = z_0$. Then the k points $z_0, z_1 = f(z_0), z_2 = f(z_1), \cdots z_{k-1} = f(z_{k-2})$ are said to form a "cycle of length k," or "k-cycle." A fixed point is then a cycle of length 1.

Cycles, like fixed points, can be attractive or repulsive, depending on whether iterations of the formula bring nearby points closer to the cycle, or push them further away. For an example of a repulsive cycle, consider the values $\dfrac{-1 + i\sqrt{3}}{2}$ and $\dfrac{-1 - i\sqrt{3}}{2}$. (These values are cube roots of 1.) It is easy to check, by iterating nearby points, that this pair forms a repulsive 2-cycle for the formula x^2. The formula $f(x) = x^2 - 1$ has an attractive 2-cycle, consisting of the points 0 and -1.

There is a simple rule for determining whether a k-cycle is attractive or repulsive, based on the value of the complex derivative. We compute the absolute value of the complex derivative of the iterate $f^{(k)}$, and evaluate it any point of the cycle. If the result is less than 1, the cycle is attractive. If it is greater than 1, the cycle is repulsive. If the result is exactly 1, the cycle is neither attractive nor repulsive, and will be called "indifferent."

If a formula has an attractive cycle, we can ask what points in the plane are attracted by the cycle. In other words, what points eventually get arbitrarily close to points in the cycle under repeated iterations of the formula. The set of these points is called the "basin of attraction" of the cycle, and can be likened to the drainage basin associated with a depression in the earth's surface.

It is easy to use a computer to find basins of attraction. Fig. 2 illustrates the basin of attraction associated with the cycle $\{0, -1\}$ for the formula $x^2 - 1$. This picture displays a portion of the complex plane centered at the origin, of diameter 4. The red and yellow portions of the picture comprise the basin of attraction to the cycle. Red corresponds to points that approach 0 on even iterations, yellow points approach 0 after an odd number of iterations. The remaining (black) portion of the picture consists of points that are attracted to infinity.

Invariance

Another concept of some importance in describing long-term behavior of a dynamical system is that of "invariance." We say that a set S of complex numbers is "invariant under f" if $f(S) = S$. For example, a cycle or a fixed point is an invariant set. If an invariant set also satisfies $f^{-1}(S) = S$, then we say that S is "doubly invariant." For example, the point 0 is doubly invariant for the function x^2, but the point 1 is not. The circle of radius 1, its interior, and its exterior are also doubly invariant sets for x^2.

What is a Julia set?

What is perhaps most interesting about Fig. 2 is not the basin of attraction, or yellow and red portion of the figure but rather the boundary between that basin and the black region. This boundary is known as the Julia set (or Julia-Fatou set) of the formula $x^2 - 1$. Julia sets are named after the mathematician Gaston Julia, who in the early 20th century elaborated many of their properties. The Julia set is a geometric object in the complex plane associated with a formula; in this sense, a picture of the formula. Any polynomial has a Julia set, and many other formulas have them, too. Like the picture in Fig. 2, most Julia sets are fractals [5], displaying an endless cascade of repeated detail.

There are several equally useful definitions of the Julia set of a polynomial, two of which we shall present here because they can be directly translated into computer algorithms. For a complete discussion, see [1].

1. The Julia set is the closure of the set of repulsive cycles of the polynomial.
2. The Julia set is the boundary between the set of points that are attracted to infinity, and the set of points that are not attracted to infinity.

How are Julia sets computed?

Using the above two definitions we provide two easy algorithms for making computer graphics pictures of the Julia set. An algorithm based on the first definition can be found in [6]:

Start with a polynomial $p(x)$ and a starting point z_0 in the complex plane, determine a sequence of complex numbers as follows:

1. Given z_n, find the inverse of $p(x)$ applied to z_n; i.e., find the solutions x of the equation $p(x) = z_n$. There will be no more solutions than the degree of the polynomial; and finding the solutions is particularly easy if p is quadratic, requiring a complex square root.
2. Choose z_{n+1} randomly from the set of solutions.

The above algorithm produces a sequence of points $\{z_n\}$ in the complex plane which converges to the Julia set. If many (thousands) of the points after the first 20 or so are plotted as separate dots on a screen, they will be seen to trace out a fractal. Fig. 5 was computed using this algorithm.

A second algorithm, based on the second definition, will more clearly delineate the Julia set for some polynomials. The object is to identify the Julia set as the boundary between points that are attracted to infinity, and points that do not converge to infinity. For example, if the polynomial has an attractive cycle, then the points that are attracted to the cycle can be rapidly determined as not converging to infinity. Different colors can be used to distinguish the two cases, thereby representing the Julia set as the boundary between two colors in a picture. To present this algorithm, assume the polynomial $p(z)$ has an attractive cycle consisting of k points, $x_1, x_2, \cdots x_n$. We shall also need to choose a "large" number N and a "small" number ϵ. Large means, in this case, large enough that for $|z| > N$, $|p(z)| > |z|$. In other words, N is large enough that numbers as large as N are all attracted to infinity. Similarly ϵ must be small enough that if z is within ϵ distance of x_1, then z will eventually converge to the cycle. Choosing numbers N and ϵ is not difficult, but may occasionally involve some trial and error.

The algorithm consists of evaluating each point on a square grid of the same resolution as the desired display device. If a point iterates to infinity, the corresponding pixel is given one color; if the point ends up in the cycle the pixel is given another color. The mapping between display device and screen coordinates must be chosen so that the desired portion of the complex plane is mapped to the screen.

For each pixel on the display device:

1. Compute the complex number $z = z_0$ that maps to the center of the pixel.
2. Iterate the function (say 50 or more times) as follows
   - compute the next iterate, $z_n = p(z_{n-1})$.
   - Check if z_n has absolute value greater than N. If so, terminate the iteration, and color the pixel appropriately.
   - Check if z_n is within distance ϵ of x_1. If so, terminate the iteration and assign the "other" color to the pixel.
3. If all 50+ iterations complete without exceeding N, color the pixel as a point that does not get attracted to infinity.

There are many variations on this algorithm which can reduce the computation, or make the resulting pictures more informative. For example, in Fig. 2 the points that do not go to infinity are recognized by the fact that they become close to elements of the cycle. If a point ends up in the cycle, the number of iterations modulo the length of the cycle can be used to color-code the various components of the basin of attraction. This is used to obtain the pattern of four colors in Fig. 3. Much of the striking use of colors in [7] results from color-coding the number of iterations required to get close to an attractive cycle.

Why are Julia sets fractals?

First, we provide an informal definition of fractal [5], sufficient for our purposes: A fractal is a geometric shape that possesses detail at all scales of magnification. In other words, one can magnify a fractal repeatedly, and more detail will appear with each magnification. Most but not all Julia sets are fractals; for example x^2 and $x^2 - 2$ have respectively a circle and straight line segment as Julia sets.

The repetitive structure of Julia sets can be explained by considering their invariance properties. If x is in the Julia set of a function f, then so is $f(x)$; conversely, if $f(x)$ is in the Julia set of f, so is x. (This follows from either of the above definitions of Julia set, and we leave it as an exercise for the reader.)

Now, given that the Julia set is invariant, suppose there is some feature F or shape that occurs in the Julia set. The double invariance of the Julia set implies that $f(F)$, the image of F under f, also lies in the Julia set. Furthermore, the conformality of the mapping f implies that $f(F)$ will appear very similar to F.

Repeatedly applying the function f, we see that any feature that occurs in the Julia set will occur again and again, distorted and rotated but of similar appearance. Not only will an infinite number of copies of such features reappear on the Julia set, but they will reappear in arbitrarily small size, everywhere along the Julia set, because the function f is expanding along the Julia set.

Many (but not all) fractals are self-similar, so that the fractal contains repeated scaled-down copies of itself. The fractals that occur as Julia sets are only approximately self-similar, in the sense that repeated structures will be distorted rather than precise magnifications of the original. This approximate self-similarity was proved by Sullivan. (A proof can be found in [1].)

Classification of Julia sets using the Mandelbrot set

Because different quadratic polynomials give rise to very different Julia sets, it is useful to have a classification of the different possible shapes that can arise. This classification is best described with the aid of another computer-generated illustration, shown in Fig. 4. This set, known as the *Mandelbrot set* (see [6], where it was first described) provides the mathematical equivalent of a road map for the space of quadratic polynomials. The Mandelbrot set should be regarded as a picture of the set of all quadratic polynomials, in the same sense that a Julia set is a picture of one particular polynomial.

For each complex number c, let $f_c(x)$ denote the polynomial $x^2 + c$. The Mandelbrot set is defined as the set of values c for which the successive iterates of 0 under f_c do not converge to infinity. It is the set of complex numbers c such that

$$\overline{\lim_{n\to\infty}} |f_c^{(n)}(0)| < \infty .$$

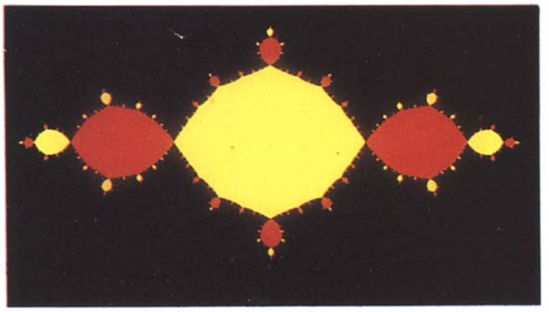


Fig. 2. This illustrates the Julia set of the formula $x^2 - 1$, which has an attractive two cycle. Points colored red are attracted to 0 on odd iterations of the formula, and points colored yellow are attracted to 0 on even iterations. The region attracted to infinity is black.

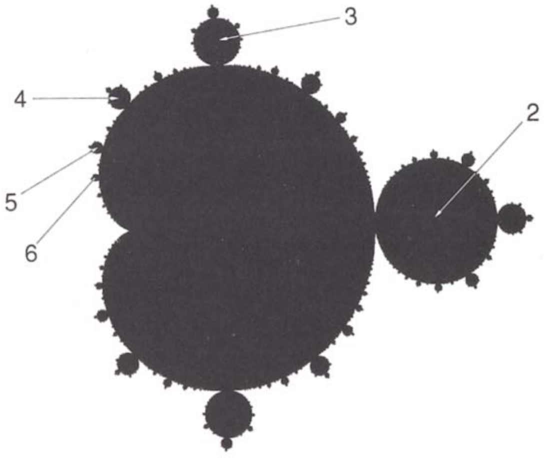


Fig. 4. This illustrates the Mandelbrot set associated with the formula $x^2 - c$. The formula is iterated for various values of c, and the point c is colored black or white depending on whether or not the iteration remains finite. The values of c designated 2, 3, 4, 5, and 6 determine formulas with attractive cycles of the respective lengths.

One easy way to compute the Mandelbrot set is as follows:

- For each complex number c on a grid, compute the iterates $f_c(0), f_c(f_c(0)), \cdots$
- If the iterate becomes large in absolute value (say greater than 5) then the point c is outside the Mandelbrot set and is shown white. Otherwise, stop the iteration after a suitable number of tries, and display the corresponding pixel as black.

As with the Julia set, more information (and more artistic license) can be obtained by color-coding the number of iterations used to determine the fate of a given point.

How the Mandelbrot set works

By looking at where a given complex number c occurs relative to the Mandelbrot set, it is possible to determine the dynamics associated with the formula $x^2 + c$, as well as to predict general properties of the Julia set of f_c. The different types of dynamics that occur when f_c is iterated are described as follows:

- The white portion of the figure, consisting of c's for which 0 iterates to infinity, corresponds to Julia sets which are Cantor sets, totally disconnected fractals. All points in the complement of the Julia set are attracted to infinity. Fig. 5 shows an illustration of this.
- The interior of the black region consists of values of c for which f_c has an attractive cycle other than infinity. The various components of the interior of the black region correspond to different cycle lengths. The central (cusped) component contains c's such that f_c has an attractive fixed point. Different com-

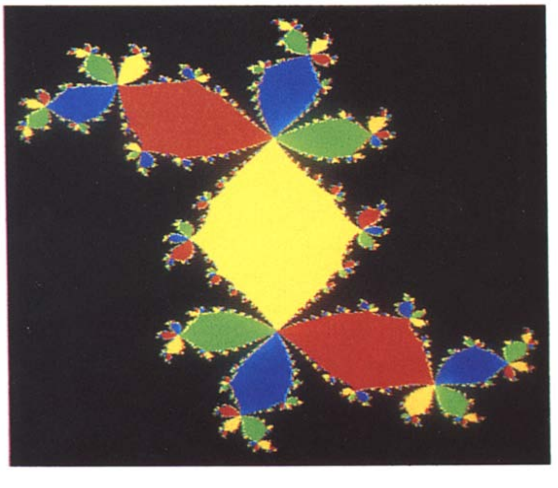
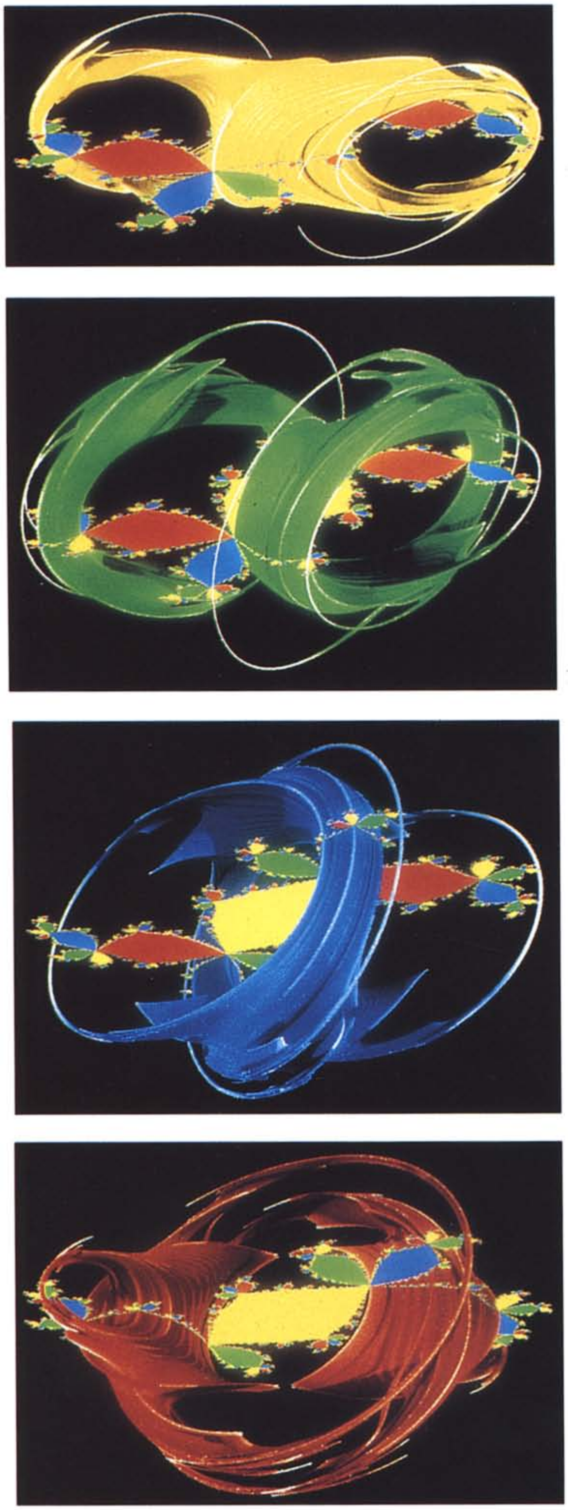


Fig. 3. This is the Julia set of the formula $x^2 + .2809 - .53i$. There is an attractive four cycle, and the four colors are used to identify the components of the basin of attraction to the cycle. Compare figures 6, 7, 8, and 9 of the quaternion Julia set of the same formula.

Fig. 5. This illustrates a totally disconnected Cantor set that is the Julia set of a quadratic polynomial. All points in the complex plane, except dots colored black, iterate to infinity.



Figs. 6, 7, 8, and 9. These four illustrations show components of the Julia set of $x^2 + .2809 - .53i$ in a three-dimensional subspace of the quaternions. The planar fractal shows the Julia set in the complex plane. The different figures show how the four colored regions extend into higher dimensions.

ponents correspond to different cyclic structures, both in the length of the cycle, and in how the components are arranged.

For example, the value $c = 0$, which lies in the middle of the central shape. The formula f_c is just x^2, having as Julia set the circle of radius 1.

Another example, $c = -1$, corresponds to the shape illustrated in Fig. 2. In this case the point lies in the center of the black circle, denoted 2, in the center. The resulting 2-cycle is indicated by the alternating colors in Fig. 2.

In general it is easy to find cycles of any length just by picking a value of c from the interior of one of these components. The cycle length is determined by the position of the component. Consider for example the sequence of shapes marked 2, 3, 4, 5, 6 in Fig. 4. They correspond to cycles of lengths 2, 3, 4, 5 and 6. Using a graphics display with crosshair, it is easy to discover values of c for which the formula $x^2 + c$ has a desired cycle structure. Then by computing the Julia set itself, one can distinguish between other, more subtle features of the quadratic functions.

- The boundary between the black and white portions of Fig. 4 is perhaps the most significant feature of the Mandelbrot set. Values of c along this boundary are associated with many different (and beautiful) dynamical systems. Not only are such Julia sets the most complex and intricate, but the mathematics of the underlying dynamics is itself not completely understood. We shall not describe the possibilities here, but refer the reader to [7] and [1]. The animation "Dynamics of $e^{i\theta}$" [9] illustrates the transition in structure of the Julia sets that occur along that boundary.

Why does the Mandelbrot set work?

In order to understand the relevance of the Mandelbrot set to the dynamics of quadratic polynomials, we reconsider its definition. The trajectory of the point 0 under repeated applications of f_c determines whether the point c lies inside or outside of the Mandelbrot set. The significance of the starting point 0 is that it is the "critical point" of f_c; that is, the derivative $f_c'(z)$ vanishes at the point $z = 0$. The Mandelbrot set can be regarded as a diagram of the behavior of the critical point under the quadratic mapping.

Fatou and Julia were well aware of the significance of the critical point in classifying the dynamics of iteration. A basic result of Fatou (see [1]) is that the basin of attraction of an attractive cycle always contains a critical point. Consequently, any attractive cycle for f_c can be found by following the iteration that starts at 0. If that iteration goes to infinity, f_c can have no attractive cycle other than infinity.

In general, the critical points of a mapping are important in describing the dynamics. Changes in the behavior of the dynamical system defined by a formula are accompanied by changes in the behavior of critical points under iteration. This general principle can be applied to many transformations other than quadratic mapping, and we shall soon see its usefulness in describing the geometric structures that occur in iteration of quaternion mappings.

Fig. 10. This picture is associated with the formula $1.06ix(1 - x)$. The planar Julia set of this formula is identical with the Julia set of $x^2 + .2809 - .53i$, illustrated in Fig. 3; except that it is rotated by 90 degrees. The four colors are used to illustrate all the components of the quaternion Julia set and how they connect to the planar fractal. The resulting quaternion Julia set is quite different topologically from the one illustrated in Figs. 6–10 because of different interconnections resulting from the placement of the imaginary axis.

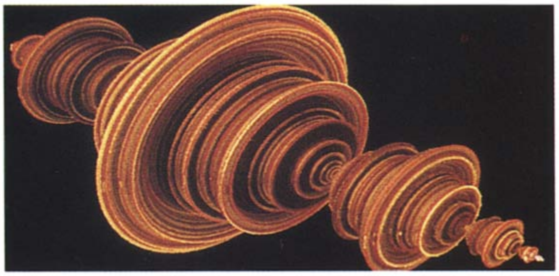


Fig. 12. This illustrates the three-dimensional extension of the Julia set of Fig. 2, from the formula $x^2 - 1$. Note that this is just the three-dimensional figure swept out by rotating the Julia set of Fig. 2 about the real axis.

EXTENDING TO HIGHER DIMENSIONS

Given the complexity and beauty of Julia sets and the Mandelbrot set, it is natural to seek higher dimensional generalizations. This is not only an interesting mathematical question; such dynamical systems can provide useful models for other disciplines, like computer graphics and physics.

In the following we show how the dynamics of the quadratic mapping gives rise to interesting geometric structures in the 4-dimensional quaternion algebra. We present these results not to show a completed analysis, but to demonstrate the wide range of possibilities for computer graphics illustration of dynamical systems.

What is a quaternion?

Quaternions were discovered in 1843 by the Irish physicist and mathematician William R. Hamilton[2]. Attempting to define a 3-dimensional multiplication, he found it necessary instead to extend to four dimensions. After this discovery, Hamilton and his contemporaries devoted a considerable effort advocating the application of quaternions to physics and other disciplines[4].

Fig. 11. This shape is also associated with a rotation of the Julia set of Fig. 3, in this case a rotation of about 30 degrees. The formula is $(2. + 1.06i)x(1 - x)$. Note that the interconnection pattern of loops differs from the pattern in Figs. 6–10.

There are several reasons for elevating quaternions above the status of "a mathematical curiosity." For example, the real numbers, the complex numbers, and the quaternions are now known (by a theorem of Hurwitz) to be the only Euclidean spaces in which we can perform addition, subtraction, multiplication, and division (by nonzero elements). The standard vector operators (dot and cross products) are naturally embedded in the multiplication of quaternions. The dynamics of motion in 3-space is thereby easily expressible in terms of quaternion operation. This relationship implies computational advantages in using quaternions to express 3-D spatial manipulation (see [10]).

We can describe quaternions as an extension of the complex plane, comparable to the previous discussion of complex numbers as an extension of the real number line. Complex numbers provide an extension of the notion of "number" to permit us to consider numbers as two-dimensional quantities. In other words, complex numbers are just a set of rules for multiplying and adding points in two dimensions. Similarly, quaternions may be regarded as a way of extending the notion of "number" to four dimensions: The rules of quaternion multiplication and addition provide a way of doing arithmetic on four-dimensional quantities.

To explicitly define quaternion multiplication, we represent points in four dimensions in the form:

$$Q = a_0 + a_1 i + a_2 j + a_3 k$$

where i, j, k are (like the imaginary number i) unit vectors in three orthogonal directions, perpendicular to the real x-axis.

To add or multiply two quaternions, we treat them as polynomials in i, j, k, but use the following rules to deal with products:

$$i^2 = j^2 = k^2 = -1;$$

$$ij = -ji = k; \quad jk = -kj = i; \quad ki = -ik = j.$$

Figs. 13, 14, and 15. These show the result when a 90 degree rotation is applied to the formula of Fig. 2, resulting in $i(x^2 + 1)$. No longer is the quaternion Julia set simply a rotation of the planar Julia set. Figs. 14 and 15 illustrate the extensions of the red and yellow components into three dimensions. Fig. 16 illustrates the combination of both red and yellow, showing how the pair of shapes is linked together infinitely many times.

We note that, since for example $ij = -ji$, the multiplication rule for quaternions is noncommutative; the result of multiplying two quaternions depends on their order. This greatly complicates the rules for doing algebra with quaternions.

Analogous to the absolute value of a complex number, we have the "norm," defined by:

$$|Q|^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

This equals the squared distance from Q to the origin in four-space.

Quaternion polynomials

Note that if we identify the quaternion i with the complex number i, then the complex plane can be regarded as situated inside the quaternions. This makes it possible for us to consider any complex polynomial as a polynomial over the quaternions as well. Expressions like:

$$ax^2 + bx + c$$

are then polynomials. However, the noncommutativity of the quaternions implies that many polynomials cannot be so simply described. For example, unless the coefficients a and b are real, the above polynomial is not equal to

$$x^2a + bx + c,$$

nor

$$x^2a + xb + c,$$

nor

$$xax + bx + c,$$

etc.

For the purposes of this article, we shall consider only a small class of such quadratic polynomials, those expressible in the form $ax^2 + b$, where a and b are complex numbers. We shall see that even this class of polynomials introduces a wealth of structure not seen in the complex quadratic mapping.

What is a Julia set of a quaternion polynomial?

We begin with the observation that a quaternion polynomial can be used to define a dynamical system on the quaternions: If $p(z)$ is a quaternion polynomial, then for any quaternion q, $p(q)$ is another quaternion. With the aid of a computer, quaternion polynomials can be easily iterated, to evaluate the long-term behavior. We can still speak of attractive and repulsive cycles, basins of attraction and the like, where the notion of complex number is replaced by quaternion.

We shall generalize the (second) definition of Julia sets for complex polynomials: The Julia set of a polynomial $p(z)$ is the boundary of the set of quaternions q such that $p^{(n)}(q)$ converges to infinity as n becomes large. There is another definition of more generality, which we shall provide in the appendix, but for the purposes of computing the shapes in this article, this one definition will suffice.

Do quaternion Julia sets extend beyond the complex plane?

Consider a complex polynomial $p(z)$. It will have a Julia set J in the complex plane, and J will necessarily be contained in the quaternion Julia set of $p(z)$. But J could in fact be the entire Julia set in the quaternions as well. In other words, extending to the quaternions could provide us with nothing new. Fortunately, some Julia sets can easily be seen to extend beyond the complex plane, forming truly 4-dimensional objects.

Fig. 16. This is associated with the formula $(0.617 + 0.774i)x(1 - x)$. The repeated pattern of holes in this shape results when the imaginary axis intersects the planar Julia set multiple times.

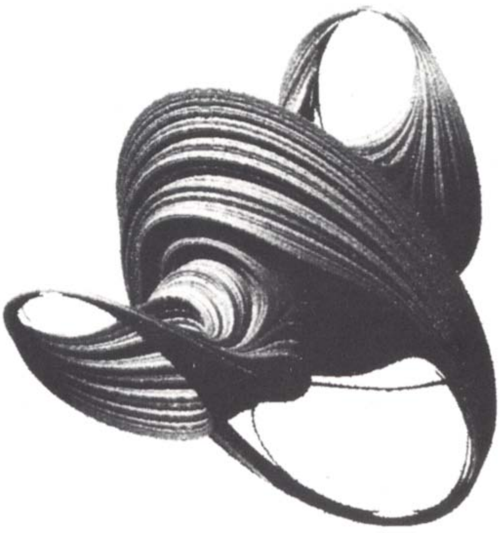


Fig. 18. This quaternionic Julia set is associated with the cubic polynomial, $x^3 + (0.596 + 0.161i)$, showing a three-fold pattern resulting from the singularity of the cubic mapping.

Consider the polynomial $p(z) = ax^2 + b$, where a and b are complex numbers, and suppose furthermore that $p(z)$ has an attractive cycle in the complex plane. That means there is an area in the complex plane of points z such that $p^{(n)}(z)$ converges into the cycle. If we show that other points outside the complex plane are also attracted into the cycle, that will imply that the domain of attraction to the cycle, as well as the Julia set, extend beyond the complex plane.

This is in fact true, and a proof will be presented in the appendix. Generally the polynomials of the form $ax^2 + b$ do have Julia sets that extend, although other polynomials, like $ax^2 + bx$, do not share this property.

How we compute and visualize a Julia set in the quaternions

Techniques for computing and viewing quaternion Julia sets were presented in [8]. We shall briefly review the methods used. The first problem is to model the Julia set in such a way as to be tractable for 3D computer graphics. The second problem is to construct a 3D image that conveys the fractal nature of the object.

How 4-dimensional quaternionic Julia sets can be computed in three dimensions

The quaternionic Julia set as defined is a subset of 4-space, and such sets can in general only be sampled, not fully exhibited in one 3D picture. However, in the particular case that the polynomial has complex coef-

Fig. 17. An illustration of $(-0.6 + 1.04i)x(1 - x)$, associated with a four-cycle. The long strands result from choosing the imaginary axis to cross multiple components of the planar Julia set.

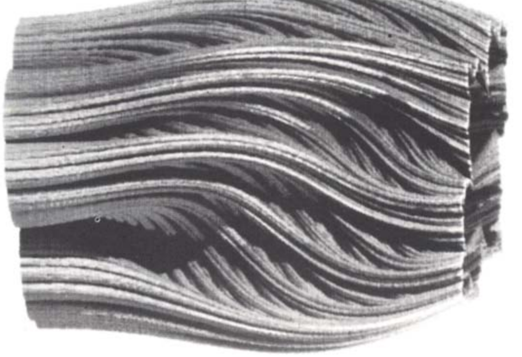


Fig. 19. A three-dimensional slice of a Mandelbrot set of $x^2 - c$, in the quaternions.

ficients we can use certain symmetries to reduce the dimension of the problem. Consider a three dimensional subspace of the quaternions that we obtain by adjoining to the complex plane *any* quaternion q that lies in the plane of j and k. The four dimensional Julia set intersects this three-dimensional space with a set J_q that depends on the q chosen. In fact, the sets J_q are congruent (i.e., of identical geometric shape) regardless of the choice of q. This means that, in order to completely understand such 4-dimensional Julia sets, we need only compute their intersection in one 3-dimensional space containing the complex plane. For example, we can use the space spanned by $1, i, j$.

Computing the 3D Julia set

In theory these could be computed by the same simple algorithm we presented above for complex Julia sets. The polynomial could be iterated, starting with each point on a 3D grid, determining for each such grid point whether the corresponding quaternion iterates to infinity. In practice, that method is intractable, requiring evaluation of billions of grid points to obtain moderately high-resolution pictures.

To compute illustrations for this article, the amount of computation is substantially reduced by only computing boundary points, rather than evaluating every vertex on the grid. The algorithm follows the boundary of the basin of attraction for a cycle, wherever the boundary may lead, tracking it through a 3-dimensional grid. Points far from the boundary are never evaluated.

By selectively following various components of the basin of attraction, it is possible to determine how these components are connected, without determining the whole Julia set. Figs. 6–9 show different components of the basin of attraction to the four-cycle associated with $p(x) = x^2 + (0.2809 - 0.53i)$. By computing and viewing them separately we determine how they are interconnected.

Making pictures of 3D fractals

Once the Julia set has been computed, as a set of vertices in a three-dimensional grid, a second computation is required to produce a two-dimensional image, suitable for raster display. Because of the fractal nature of the surface it is important to present the image in a manner that conveys information about surface texture. When we view real physical objects, we understand the three-dimensional surface structure by noticing shadows and shading on the surfaces. Similar visual cues must be generated by the computer if we are to perceive the surface structure.

In the images presented here, we simulate the surface illumination by a "z-buffer" algorithm [8]: All surface elements (i.e., grid points) are projected into a depth buffer to determine the element closest to the light source. Only surfaces visible to the light source receive illumination. Only after the illumination is complete, and a brightness value is known for each surface element, is the actual image computed. This is done through another z-buffer projection to the viewer.

Interconnections and loops in Julia sets

The images of quaternionic Julia sets, resulting from the above computation, reveal a surprising wealth of detail, not obviously deducible from the Julia sets in the complex plane. Perhaps the most striking features of these shapes are the long strands or loops that interconnect different portions of the complex Julia set.

Consider for example the shape depicted in Fig. 6. This shows the complex Julia set as a planar slice through the quaternionic Julia set. Only one component of the basin of attraction to the four-cycle has been followed into the quaternions. We see how some of the planar components (colored yellow) become interconnected in three dimensions, and others do not.

We can alter the loop interconnection pattern (and therefore the quaternionic Julia set) without changing the complex planar Julia set. For any angle θ, the formula $e^{i\theta}x^2 + e^{-i\theta}c$ defines a Julia set in the complex plane that is congruent to the Julia set of $x^2 + c$. This just rotates the Julia set of $x^2 + c$ by the angle θ about the origin in the complex plane. This change of the formula does more than just a rotation in the quaternions, resulting in a change in the topological structure of the Julia set. See for example Figs. 10 and 11, illustrating the quaternionic Julia sets that result when the angle θ is, respectively, $\frac{\pi}{2}$ and $\frac{\pi}{6}$ radians.

We shall show how one can predict these interconnections, and provide a mechanism for obtaining a set with desired connections. To explain the interconnection patterns it is necessary to discuss the squaring mapping $p(x) = x^2$ in terms of how it acts on the 4-dimensional space of quaternions.

The squaring mapping and its critical set in the quaternions

Recall first how the squaring function acts on the complex plane. If we represent a complex number in polar coordinates, $z = re^{i\theta}$, then its square is $z^2 = r^2e^{2i\theta}$. Squaring results in a doubling of the polar angle, so that the mapping wraps the plane twice around the origin. Squaring is a two-to-one mapping except at the single critical point, 0, which is mapped to itself.

Similarly, in the quaternions, the squaring map is usually a two-to-one map: A quaternion and its negative have the same square. However, there is a much larger critical set, where the mapping fails to be two-to-one. Recall from the definition of quaternion multiplication that $i^2 = j^2 = k^2 = -1$. This shows that -1, at least, has more than two square roots. In fact, there is an entire two-dimensional sphere of quaternions q such that $q^2 = -1$; namely q can be any norm-one quaternion with zero real part.

The critical set of the quaternion polynomial $ax^2 + b$ is precisely the set of quaternions of zero real part. The above examples illustrate the fact that the quaternion squaring map fails to be two-to-one precisely on that set of quaternions of zero real part; the square of any such quaternion lies on the negative real axis.

We describe the squaring mapping as follows: Let U be the set of quaternions with positive real part, and $-U$ those with negative real part. Under a squaring operation, both U and $-U$ are mapped onto the complement of the negative real axis. Every quaternion, except those on the negative real axis, has exactly two square roots, one in U and one in $-U$. The boundary of U (the quaternions with zero real part) is folded onto the negative real axis, taking all points of the boundary of radius r to the negative real number $-r^2$.

Collapsing of two-spheres

Note that the inverse under the squaring map of any negative real number is a two-dimensional sphere, so that the action of the mapping is to collapse such spheres to points. This "collapse of 2-spheres" has consequences in the geometry of shapes invariant under a squaring operation. Suppose a set S is invariant under the polynomial $ax^2 + b$. If a point q of S lies in the set which is collapsed to a point under the squaring, then invariance of S implies that the entire collapsing 2-sphere containing q lies in S. Since the Julia set is an example of an invariant set, this implies the existence of numerous 2-spheres in the Julia set, if the Julia set crosses the set of quaternions with zero real part.

The loops in the 3-dimensional shapes pictured here can be explained by this phenomenon. The 2-spheres in the Julia set correspond to 1-spheres (loops) in the 3-dimensional slices of the Julia set that we are displaying. The loops that are visible in the illustrations result from intersections between the Julia set in the complex plane and the imaginary axis: If such an intersection occurs at a point P in the complex plane, then there is a loop in the 3D shape, connecting P and $-P$ in a circle. Once one such loop L exists, there will occur in the Julia set an infinite cascade of other loops, namely $f^{-1}(L), f^{-1}(f^{-1}(L)), \ldots$, etc.

How can we determine the topology of a quaternionic Julia set?

The discussion above gives a recipe for constructing loops in the quaternionic Julia set. More generally, we can design Julia sets with various interconnection patterns.

Consider again the formula $f_c = x^2 + c$, having complex planar Julia set J_c, and suppose that f_c has an attractive cycle, so that J_c is the boundary between the points (in the basin of attraction) that are attracted to the cycle, and the points attracted to infinity. If two components of the planar basin of attraction are located diametrically opposite, we can modify the formula to cause those components to be interconnected by a loop in the quaternions.

This is done as follows: Choose a line through the origin that intersects both the opposite components. Let θ be the angle of rotation (clockwise) between the positive x-axis and the line. The formula to be iterated in the quaternions is then $e^{i\theta}x^2 + e^{-i\theta}c$. This formula then has the same complex planar Julia set as f_c, except that it is rotated by the angle θ about the origin. In the quaternions, however, the two components to be connected now lie on the imaginary axis, so there will be a loop (more precisely, a two-sphere in four-space) in the quaternion basin of attraction, intersecting and connecting them both.

In some instances, not only one loop, but an infinite cascade of interconnected loops are created by this process. Suppose for example that a component of the 2-D basin of attraction that intersects the loop contains a point of the attractive cycle (not just a preimage of the cyclic point). The connected 4-dimensional component of that cyclic point will then be an invariant set under the mapping f. The invariance of this component implies that it will contain an infinite number of loops, preimages of the starting loop.

One final example to illustrate this technique. We have seen already (in Fig. 2) the complex planar Julia set of $x^2 - 1$. When computed in the quaternions Fig. 12 results, a surface of revolution about the x axis. The only interconnections are those introduced by the components on the imaginary axis, and all of the loops are simple rings about the x axis.

Instead, let us rotate the complex Julia set by 90 degrees, obtaining the formula $i(x^2 + 1)$. The components of the basin of attraction, formerly situated along the real axis at -1 and $+1$, are now along the imaginary axis at $+i$ and $-i$. This causes the component containing $-i$ and its opposite, the component containing i, to become connected in the quaternions. By considering the preimages of this pair (these are necessarily connected as well), it is not difficult to show that all of the quaternionic basin of attraction becomes divided into just two connected components, illustrated in Figs. 13 and 14. The red component consists of points attracted to 0 on odd iterations, and the yellow points are attracted to 0 on even iterations. These two components fit together (and intertwine) without overlap, as illustrated in Fig. 15.

Other directions

We have only presented some of the simplest geometric manipulations that can be imposed on the quaternion Julia sets. Figs. 16–18 illustrate some of the further directions that one can explore. These depict the formulas $(0.617 + 0.774i)x(1 - x)$, $(-0.6 + 1.04i)x(1 - x)$, and $x^3 + (0.596 + 0.261i)$, respectively. Fig. 16 shows that, by careful placement of the critical set, it is possible to cause repeating patterns of holes in the Julia set (rather than repeating patterns of interconnectivity). Fig. 17 demonstrates how long strands in the quaternionic Julia set can be induced by appropriate manipulation of a "stringy" Julia set in the complex plane. Fig. 18 shows the Julia set of a cubic polynomial, illustrating that the critical set of higher-degree polynomials can similarly be manipulated.

Mandelbrot sets in higher dimensions

Implicit in the above discussion (but not proved here) is a method for describing the structure of quaternionic Julia sets for quadratics of the form ax^2

+ b, with a and b complex numbers. That structure can be derived from a knowledge of the planar Julia set together with a parameter θ, determining how the imaginary axis (and therefore the critical set) intersects the planar Julia set. A three-dimensional space (the planar Mandelbrot set, together with θ) is required to parameterize these dynamical systems. There is a space that serves as Mandelbrot set for these functions, classifying the dynamics that occurs in the quaternions. Unfortunately, that space can be embedded only in four dimensions, so we can only show three-dimensional slices as exhibited in Fig. 19. This Mandelbrot set can be identified with the boundary of the set of pairs of complex numbers (z, c) such that $f_c^{(n)}(z)$ remains bounded as $n \to \infty$. The slice of this shape in Fig. 19 is the subset defined by restricting c to values on the boundary of the central component of the Mandelbrot set.

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APPENDIX

In this section we provide a mathematical justification for some of the statements in the paper. The mathematical level of this discussion assumes understanding of advanced calculus.

Formal definition of Julia sets

The usual definition of Julia set is different than the ones we have used in this article. We begin by recalling the notion of **equicontinuity**: A set of functions $\{f_i\}$ is said to be equicontinuous at a point x if for any constant δ there is an ϵ such that for every i

$$|f_i(x + h) - f_i(x)| < \delta$$

whenever $|h| < \epsilon$.

This condition is stronger than saying that all the functions f_i are continuous at x, because it requires the same ϵ to work for all the functions. If all the derivatives of the functions f_i are bounded by the same constant, then the set of functions is equicontinuous.

The Julia set of a complex function $f(x)$ is the set of complex numbers z such that the set of all iterates of f, $\{f^{(i)}\}$, is not equicontinuous at z. For quaternion functions we take the same definition, replacing absolute values in the above statement with the quaternion norm.

This definition does not necessarily coincide with the one we used above, unless some restrictions are placed on the functions f. If f is a complex polynomial then Montel's theorem can be used to show the equivalence (see [1]). If f is a quaternion polynomial the equivalence is not clear. It may occur that for some quaternion polynomials the boundary of the set of points attracted to infinity is a proper subset of the Julia set, although this has not been observed.

Derivatives of quaternion polynomials

To reason about attraction and repulsion in the quaternions it is useful to have a means of computing derivatives. Because quaternion polynomials are mappings on four-dimensional space, their derivatives cannot be represented simply as a numerical value. The derivative of such a function $f(x)$ at a point x must be regarded instead as the Jacobian matrix, or the linear transformation that best approximates the polynomial near the point x. In other words, the derivative T of f at x is a linear transformation such that $f(x + y)$ is approximately equal to $f(x) + T(y)$ for y near 0.

An easy way of representing this linear transformation is to specify what it does to a quaternion y. The derivative of a polynomial is the sum of the derivatives of the monomials in it, so it is enough to differentiate monomials. The derivative of a monomial, evaluated at y, is the sum of all the monomials that are each obtained by replacing one occurrence of x by the value y. Some examples: The derivative of x^2 at a quaternion x is the linear transformation that takes a quaternion y to the quaternion $xy + yx$. The derivative of x^3 is the transformation whose value at y is $x^2y + xyx + yx^2$.

Attractive cycles in the quaternions

We can use the above formulation of the Jacobian derivative to show that attractive cycles for complex polynomials of the form $ax^2 + b$ extend to be attractive in four dimensions, implying that the quaternion Julia sets of such polynomials are nontrivial extensions of their complex planar counterparts. It suffices to consider an attractive fixed point p; cycles admit a similar proof. If p is an attractive fixed point, then we know the absolute value of the complex derivative satisfies $|f'(p)| < 1$. To show that p is attractive in the quaternions, we need to know that Jacobian matrix of f at p, as a linear transformation, has norm less than one. The Jacobian of $ax^2 + b$, applied at p to a quaternion y has value $a(py + yp)$. Using the multiplicativity of the quaternion norm, we see that the norm is at most $2|a||p||y|$. Since $2|a||p|$ is the absolute value of the complex derivative of f at p, we conclude that the Jacobian matrix does in fact have norm less than one.